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# A $C^1$ FUNCTION FOR WHICH THE $\omega$-LIMIT POINTS ARE NOT CONTAINED IN THE CLOSURE OF THE PERIODIC POINTS

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ABSTRACT. We develop a $C^1$ function $f : [-\frac{1}{6}, 1] \to [-\frac{1}{6}, 1]$ for which $\Lambda(f) \neq \overline{P(f)}$. This answers a query from Block and Coppel (1992).

## 1. Introduction

Let $I$ be the unit interval of the real line. By $\mathcal{C}$ we denote the space of all continuous maps of $[0, 1]$ into itself, and, for $r \geq 1$, by $\mathcal{C}^r$ we denote the space of all maps of $[0, 1]$ into itself having all the derivatives continuous up to the $r$th one. Let $f : I \to I$ be a continuous map of the interval into itself. Having performed the map $f$ once, one can perform it again, and again, and again. That is, we consider the *iterates* $f^n$ defined inductively by

$$f^1 = f, \quad f^{n+1} = f \circ f^n \quad (n \geq 1).$$

We also take $f^0$ to be the identity map, defined by $f^0(x) = x$ for every $x \in I$.

A subset $J$ of $I$ is *periodic* with period $n$ if $f^n(J) = J$ and $f^i(J) \neq J$ for $1 \leq i \leq n-1$; and $Orb(J) = J \cup f(J) \cup \ldots \cup f^{n-1}(J)$ is the *orbit* of $J$. By $P(f)$ we denote the set of all periodic points of $f$ and by $\overline{P(f)}$ its closure.

We define the *trajectory* of a point $x \in I$ to be the set $\gamma(x) = \{f^n(x) : n \geq 0\}$.

We define the *$\omega$-limit set* of a point $x \in I$ to be the set

$$\omega(x) = \omega(x, f) = \bigcap_{m \geq 0} \overline{\left(\bigcup_{n \geq m} f^n(x)\right)}.$$

Evidently, $y \in \omega(x)$ if and only if $y$ is a limit point of the trajectory $\gamma(x)$; that is, $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to +\infty$.

By $\Lambda(f)$ we denote the set

$$\Lambda(f) = \bigcup_{x \in I} \omega(x).$$

A point $x \in I$ is said to be *recurrent* if $x \in \omega(x)$ and *non-wandering* if every open set containing $x$ contains at least two points of some trajectory. Equivalently, $x$ is recurrent if $f^{n_k}(x) \to x$ for some sequence of integers $n_k \to +\infty$, and non-wandering if $f^{n_k}(x_k) \to x$ for some sequence of points $x_k \to x$ and some sequence of integers $n_k \to +\infty$. Let $R(f)$ and $\Omega(f)$ denote respectively the set of all recurrent and non-wandering points of $f$. For the properties of these sets we refer to [1].

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It follows at once from the definitions that

$$f(P(f)) = P(f) \subseteq R(f) = f(R(f))$$

and

$$R(f) \subseteq \Lambda(f) \subseteq f(\Omega(f)) \subseteq \Omega(f) = \overline{\Omega(f)}.$$

Examples show that each of the inclusions here can be strict.

Sharkovskii [13] has shown that $\Lambda(f)$ is closed, and hence $\overline{P(f)} \subseteq \Lambda(f)$ ([1], page 77), and Nitecki [10] has shown that if $f$ is piecewise monotone, then $\Lambda(f) = \overline{P(f)}$. Moreover, for every non-chaotic map $f \in \mathcal{C}$, $\Lambda(f) = \overline{P(f)}$ ([4], Theorem 2.1).

Coven, Madden and Nitecki ([5], the Permanence Lemma) proved that $\Omega(f) = \overline{P(f)}$ generically in $\mathcal{C}$ and hence, from the inequality $\overline{P(f)} \subseteq \Lambda(f) \subseteq \Omega(f)$, it follows that $\Lambda(f) = \overline{P(f)}$ generically in $\mathcal{C}$.

Jakobson [9] has studied endomorphisms of the circle and he proved that $\overline{P(f)} = \Omega(f)$ for an open dense subset of $End^1(S^1)$ (and hence $\mathcal{C}^1$), where $S^1$ is the unit circle and $End^1(S^1) = \{f : S^1 \to S^1$ continuous with its first derivative$\}$. Hence $\Lambda(f) = \overline{P(f)}$ for every $f$ of an open dense subset of $\mathcal{C}^1$.

Young [18] has shown that $\overline{P(f)} = \Omega(f)$ for all elements of an open dense subset of $\mathcal{C}^r$, $2 \le r \le \infty$, and hence it follows that $\Lambda(f) = \overline{P(f)}$ for all elements of an open dense set of $\mathcal{C}^r$, $2 \le r \le \infty$. So, putting together Jakobson's Theorem, Theorem 2 in [18], and Pugh's proof in [11], we obtain that in $\mathcal{C}^r$, $1 \le r \le \infty$, $\Lambda(f) = \overline{P(f)}$ generically. In [2] Block and Coven construct an example of a map $f \in \mathcal{C} \setminus \mathcal{C}^1$ with zero topological entropy for which $\overline{P(f)} \neq \Lambda(f)$.

We answer the following open problem ([1], page 81):

**Question A.** Is $\overline{P(f)} = \Lambda(f)$ for every $f \in \mathcal{C}^1$?

**Definition 1.1** ([15], page 740)**.** For a given $f \in \mathcal{C}$, consider the system $\{\omega(x) : x \in I\}$ ordered by inclusion. Following Sharkovskii, a *maximal $\omega$-limit set* of $f$ is *of the second kind* if it contains a periodic point and is infinite; otherwise, it is a *maximal $\omega$-limit set* of $f$ *of the first kind*. Denote by $A_1(f)$ and $A_2(f)$ the classes of infinite maximal $\omega$-limit sets of $f$ of the first and second kind, respectively.

**Theorem 1.2** ([15], Theorem 3.7 (v))**.** *Let $f \in \mathcal{C}$. If $\omega(x) \in A_2(f)$, then the set $\omega(x) \cap P(f)$ is dense in $\omega(x)$.*

**Theorem 1.3** ([14])**.** *Let $f \in \mathcal{C}$. Every infinite $\omega(x) \in A_1(f)$ is of type $Q_{\omega(x)} \cup N_{\omega(x)}$, where*

1. *$Q_{\omega(x)}$ is a Cantor set,*
2. *$N_{\omega(x)}$ is countable, disjoint from $Q_{\omega(x)}$, dense in $\omega(x)$ if nonempty, and each point of $N_{\omega(x)}$ is isolated, and*
3. *every interval contiguous to $Q_{\omega(x)}$ contains at most two points of $N_{\omega(x)}$ and each of the intervals $[0, \min Q_{\omega(x)}]$, $[\max Q_{\omega(x)}, 1]$ contains at most one point of $N_{\omega(x)}$.*

*Moreover, $Q_{\omega(x)} \subseteq \overline{P(f)}$.*

Let $f \in \mathcal{C}$. In [6] Bruckner and Ceder study the map $\omega_f : I \to \mathcal{K}$ defined so that $\omega_f(x) = \omega(x, f)$. They have shown that this map is rarely continuous, is always in the second Baire class,

$$f \text{ is non-chaotic } \Rightarrow \omega_f \text{ is Baire 1},$$

and that the reverse implication is not valid. From Theorem 3.7 in [6], Theorem 1.2 and Theorem 1.3 above, it follows that

$$\omega_f \text{ is Baire 1 } \Rightarrow \Lambda(f) = \overline{P(f)},$$

but the reverse implication is not valid as shown by Hsin and Xiong in [3].

Another problem that we answer is the following:

**Question B.** If $f \in \mathcal{C}^1$ and $h(f) = 0$, can an infinite $\omega(x) \in A_1(f)$ have isolated points?

*Remark* 1.4. We give a positive answer to Question B; and from this it follows that the answer to Question A is negative (and so we also disprove a conjecture of Sharkovskii ([14], page 137)).

We proceed through a couple of sections. In section 2 we provide some basic definitions and notation that we shall use in the sequel and record some previously known results that we shall need in the course of our construction. In section 3 we construct a function $f \in \mathcal{C}^1$ with the following properties:

(1) $f$ has zero topological entropy and a unique $2^n$ cycle for every $n$, and
(2) $f$ possesses a unique uncountable $\omega$-limit set, and this $\omega$-limit set has isolated points.

## 2. Preliminaries

Let $f \in \mathcal{C}$. If $x$ is periodic of period $n$ we say that the set $\{f(x), f^2(x), \ldots, f^n(x)\}$ is a *periodic orbit of order* $n$ or an $n$*-cycle*. We say that $f$ is a $2^n$*-function* if $f$ has cycles of order equal to each $2^k$ for $k \leq n$ and no others. We say that $f$ is a $2^\infty$*-function* if $f$ has cycles of order equal to each power of 2 and no others.

If $f$ has zero topological entropy, we write $h(f) = 0$. There are many ways of characterizing this class of functions [8]. For our purpose it is sufficient to mention only the following characterization: $h(f) = 0$ *if and only if* $f$ *is a* $2^\infty$*-function or a* $2^n$*- function for some* $n$. By $\mathcal{E}$ we denote the closed collection of all $\mathcal{C}$ functions with zero topological entropy ([17], Lemma 4.3).

An important tool in the construction of our function is the following theorem due to Smital [16]:

**Theorem 2.1.** *Let* $f : I \to I$ *be a continuous function with zero topological entropy, possessing an infinite* $\omega$*-limit set* $\Omega$*. Then there exists a sequence of closed intervals* $\{J_k\}_{k\in\mathbb{N}}$ *such that*

(1) *for each* $k$, $\{f^i(J_k)\}_{i=1}^{2^k}$ *are pairwise disjoint and* $J_k = f^{2^k}(J_k)$,
(2) *for each* $k$, $J_{k+1} \cup f^{2^k}(J_{k+1}) \subseteq J_k$,
(3) *for each* $k$, $\Omega \subseteq \bigcup_{i=1}^{2^k} f^i(J_k)$,
(4) *for each* $k$ *and* $i$, $\Omega \cap f^i(J_k) \neq \emptyset$.

We call the set $\mathcal{J}$ of all such $f^i(J_k)$ a *simple system for* $\Omega$ *relative to* $f$.

We will reverse Smital's construction of simple systems by constructing a continuously differentiable $2^\infty$-function, possessing an infinite $\omega$-limit with isolated points, from a "certain" system of intervals. In order to accomplish this project we use a device suggested in [7]. To this end, let $\mathbb{N}$ denote the set of positive integers and let $\mathcal{N}$ be the set of sequences of zeros and ones. If $\mathbf{n} \in \mathcal{N}$ and $\mathbf{n} = \{n_i\}_{i=1}^{\infty}$, we write $\mathbf{n}|k = (n_1, n_2, \ldots, n_k)$. By $\mathbf{0}$ (resp. $\mathbf{1}$) we mean that $\mathbf{n} \in \mathcal{N}$ such that

$n_i = 0$ (resp. 1) for all $i$. Now, define a function $A : \mathcal{N} \to \mathcal{N}$ by $A(\mathbf{n}) = \mathbf{n} + 1\mathbf{0}$, where the addition is modulo 2 from left to right. For example, $A(\mathbf{0}) = (1, 0, \ldots)$, $A(1, 0, \ldots) = (0, 1, 0, \ldots)$, and $A(\mathbf{1}) = \mathbf{0}$. This function $A$ is called the *adding machine*.

For each $k \in \mathbb{N}$ and $i \in \mathbb{N}$ we put

$$F_{\mathbf{1}|k} = J_k \quad \text{and} \quad F_{A^i(\mathbf{1}|k)} = f^i(J_k).$$

It follows from (1) that for any $\mathbf{m}, \mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $A^j(\mathbf{m})|k = \mathbf{n}|k$. Hence, the above relations actually define $F_{\mathbf{n}|k}$ for all $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$ so that the collection of all $F_{\mathbf{n}|k}$ coincides with the simple system $\mathcal{J}$.

Recasting (1) through (4) into the new notation we have the following:

(a) For each $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$, $F_{\mathbf{n}|k,0}$ and $F_{\mathbf{n}|k,1}$ are disjoint closed subintervals of $F_{\mathbf{n}|k}$ which are interchangeable by $f^{2^k}$.
(b) For each $k \in \mathbb{N}$, $f$ maps the collection $\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$ onto itself.
(c) For each $k \in \mathbb{N}$, $\Omega \subseteq \bigcup\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$.
(d) For each $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$, $\Omega \cap F_{\mathbf{n}|k} \neq \emptyset$.

Now put

$$K = \bigcup_{\mathbf{n} \in \mathcal{N}} \bigcap_{k=1}^{\infty} F_{\mathbf{n}|k}$$

and

$$F_{\mathbf{n}} = \bigcap_{k=1}^{\infty} F_{\mathbf{n}|k}.$$

Then $K$ and each $F_{\mathbf{n}}$ are compact. For a fixed $\mathbf{n} \in \mathcal{N}$, $\{F_{\mathbf{n}|k}\}_{k=1}^{\infty}$ is descending. Therefore, the components of $K$ consist of the $F_{\mathbf{n}}$ sets. Moreover,

$$K = \bigcap_{\mathbf{n} \in \mathbb{N}} \bigcup_{k=1}^{\infty} F_{\mathbf{n}|k} = \bigcap_{\mathbf{n} \in \mathbb{N}} \bigcup_{i=1}^{2^k} f^i(J_k).$$

Let $S$ consist of all $x$ for which there exists $\mathbf{n} \in \mathcal{N}$ such that $\{x\} = F_{\mathbf{n}}$. Let

$$Q = \bar{S} \quad \text{and} \quad C = (K \setminus \operatorname{int} K) \setminus Q.$$

Then $Q$ is a Cantor set (i.e., perfect, nowhere dense and nonvoid). Also $C$ is countable and possibly empty.

Let $G$ be the component of $[0,1] \setminus K$ that contains the interval between $F_0$ and $F_1$. In general, let $G_{\mathbf{n}|k}$ be that component of $[0,1] \setminus K$ that contains the interval between $F_{\mathbf{n}|k,1}$ and $F_{\mathbf{n}|k,0}$. Let

$$\mathcal{G} = \{G\} \cup \{G_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\},$$
$$G^0 = G \cup [0, \inf K) \cup (\sup K, 1],$$

and

$$G^k = \bigcup\{G_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}.$$

Note that $[\inf K, \sup K] = K \cup (\bigcup \mathcal{G})$ and $[0,1] = K \cup \bigcup_{j=0}^{\infty} G^j$.

**Proposition 2.2** ([6], Proposition 3.1)**.** *Let $\Omega$ be an infinite $\omega$-limit set for a $2^\infty$-function $f$ with $\mathcal{F} = \{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\}$ a simple system for $\Omega$ relative to $f$. Then:*

(1) *For each component* $F_{\mathbf{n}}$ *of* $K$, $f(F_{\mathbf{n}}) = F_{A(\mathbf{n})}$.
(2) *For each* $j$, $f^j[a,b] \cap [a,b] = \emptyset$ *for each component* $(a,b)$ *of* $\operatorname{int} K$.
(3) $\Omega \cap \operatorname{int} K = \emptyset$. *In fact,* $\operatorname{int} K$ *contains no point in any* $\omega$*-limit set.*
(4) *For some* $B \subseteq C$, $\Omega = Q \cup B$.
(5) *If* $c \in C$, *then* $c$ *is an endpoint of a component of* $\operatorname{int} K$ *and* $c$ *is isolated in* $Q \cup C$.
(6) *If* $(a,b)$ *is a component of* $\operatorname{int} K$, *then either both* $a$ *and* $b$ *are in* $Q$, *or one is in* $Q$ *and the other in* $C$.
(7) *For all* $x \in K$, $\omega(x,f) = Q$.
(8) *If* $\operatorname{int} K \neq \emptyset$, *then* $\overline{\operatorname{int} K} = K$.
(9) *If* $B \neq \emptyset$, *then* $Q \subseteq \bar{B}$.
(10) $C$ *can have at most 2 points in any component of* $[0,1] \setminus Q$ *and at most one point in* $[0, \inf Q)$ *and* $(\sup Q, 1]$.
(11) *If* $\Omega^{'}$ *is an* $\omega$*-limit set that intersects* $\Omega$, *then* $\Omega \subseteq \Omega^{'} \subseteq Q \bigcup C$.
(12) *For each* $x$ *either* $Q \subseteq \omega(x,f) \subseteq K$ *or* $\omega(x,f) \subseteq \bigcup_{j=0}^{k} G^j$ *for some* $k \in \mathbb{N}$.
(13) *If* $F_{\mathbf{n}} \cap B \neq \emptyset$, *then* $F_{\mathbf{n}} \subseteq \operatorname{int}(F_{\mathbf{n}|k})$ *for all* $k$.

Throughout, by $(\mathcal{K}, \mathcal{H})$, we denote the class of nonempty closed sets $\mathcal{K}$ contained in $[0,1]$ with the Hausdorff metric $\mathcal{H}$. It is well known that $(\mathcal{K}, \mathcal{H})$ is compact.

Given a bounded interval $J$ of the real line, by $|J|$ we denote the Lebesgue measure of $J$.

Given a function $f \in \mathcal{C}$, by $sl(f)$ we denote the slope of $f$.

## 3. Example

A brief discussion of the ideas behind our construction may prove helpful. Our intention is to reverse Smital's Theorem and let the sets $F_{\mathbf{n}|k}$ determine a first function $l$, rather than the other way around. We shall first construct a function $l$ with the following properties:

(1) $h(l) = 0$,
(2) $l$ possesses a unique uncountable $\omega$-limit set,
(3) each turning point of $l$ is isolated, and these turning points have 0 as their unique point of accumulation, and
(4) $l'(0) = 0$.

We shall construct our desired function $f$ as a modification of the uniform limit of an appropriately chosen sequence $\{l_k\}_{k=2}^{\infty}$. We will begin with a certain collection of sets coded by finite tuples of 0's and 1's with properties similar to those of simple systems and produce a $\mathcal{C}^1$ function with zero topological entropy possessing an infinite $\omega$-limit set with isolated points.

Before beginning the construction it may be helpful to outline some of its main features. We develop a sequence of functions $\{l_k\}$, each of which is a $2^k$ function with periodic intervals $\{T_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$ so that $T_{\mathbf{n}|j,1}$ always lies to the left of $T_{\mathbf{n}|j,0}$ for all $\mathbf{n} \in \mathcal{N}$ and all $0 \leq j \leq k-1$. We construct closed intervals $T_0$, $T_{10}$, $T_{110}$, $\ldots$, $T_{\mathbf{1}|(k-1),0}$, $T_{\mathbf{1}|k}$ and $l_k$ so that $l_k$ is linear on their convex hull as well as on the complementary intervals $G^k$ and $G^k_{\mathbf{1}|j}$ for $1 \leq j \leq k-1$. We take the slope of $l_k$ to be $\frac{1}{2}$ on $T_{10}$, $\frac{1}{4}$ on $T_{110}$ and, in general, $\frac{1}{2^{j-1}}$ on $T_{\mathbf{1}|(j-1),0}$ for $2 \leq j \leq k$, and $\frac{1}{2^j}$ on $T_{\mathbf{1}|j}$ for $1 \leq j \leq k$. By choosing the slopes on $T_{\mathbf{1}|(j-1),0}$ for $2 \leq j \leq k$ and on $T_{\mathbf{1}|j}$ for $1 \leq j \leq k$ in this manner, we force the slope on the $2^{k-1}$ intervals contained in the convex closure of $T_0$ to be $(2^{2^k-1})^{\frac{1}{2^{k-1}}}$, since the product of the slopes on the

$2^k$ intervals $\{T_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$ must be one. We note that $\lim_{k\to\infty} (2^{2^k-1})^{\frac{1}{2^{k-1}}} = 4$. To completely define $l_k$ on $I$ we now need only to give a length to the intervals $T_{\mathbf{1}|k}$, $G^k$ and $G^k_{\mathbf{1}|j}$ for $1 \leq j \leq k-1$; the lengths of the other periodic intervals $T_{\mathbf{n}|k}$ and the complementary intervals $G^k_{\mathbf{n}|j}$ for $1 \leq j \leq k-1$ are determined by the adding machine. When constructing each $l_k$ we take $T_{\mathbf{1}|k}$ to have length $\frac{1}{3^k}$, $G^k$ to have length $\frac{1}{3}$, $G^k_{\mathbf{1}|j}$ to have length $\frac{1}{3^{j+1}}$ for $1 \leq j \leq k$.

Before giving a precise definition for the arbitrary $l_k$ we describe $l_2$ and $l_3$ in some detail. In particular, we let the slope of $l_2$ be $\frac{1}{2^2}$ on $T_{11}$, $\frac{1}{2}$ on $T_{10}$ and $2^{\frac{3}{2}}$ on $T_{01}$, $T_{00}$ and the complementary interval $G^2_0$ between them. We let the length of $G^2$ be $\frac{1}{3}$ and the length of $G^2_{\mathbf{1}|1}$ be $\frac{1}{3^2}$ and, since $\max T_{00} < 1$, we set $l_2(x) = l_2(\max T_{00})$ for every $x \in (\max T_{00}, 1]$.

At the end of stage 2, sets $\{T_{\mathbf{n}|2} : \mathbf{n} \in \mathcal{N}\}$ and $\{G^2_{\mathbf{n}|1} : \mathbf{n} \in \mathcal{N}\}, G^2$ and a function $l_2$ have been defined with the following properties:

(1) $l_2$ is a $2^2$ piecewise linear function,
(2) $sl(l_2) = \frac{1}{2^2}$ for every $x \in T_{11}$,
(3) $sl(l_2) = \frac{1}{2}$ for every $x \in T_{10}$,
(4) $sl(l_2) = 2^{\frac{3}{2}}$ for every $x \in T_{01}$,
(5) $sl(l_2) = 2^{\frac{3}{2}}$ for every $x \in T_{00}$,
(6) $|sl(l_2)| < \frac{\frac{1}{2}\ \frac{1}{3}}{\frac{1}{3^2}}$ for every $x \in G^2_1$,
(7) $|sl(l_2)| < 3$ for every $x \in G^2$,
(8) $sl(l_2) = 2^{\frac{3}{2}}$ for every $x \in G^2_0$,
(9) the length of $T_{11}$ is $\frac{1}{3^2}$, the length of $G^2_1$ is $\frac{1}{3^2}$.

To develop $l_3$ we let the slope of $l_3$ be $\frac{1}{2^3}$ on $T_{111}$, $\frac{1}{2^2}$ on $T_{110}$, $\frac{1}{2}$ on $T_{100}$ and $T_{101}$ as well as on the complementary interval $G^3_{10}$ between them, and $2^{\frac{7}{4}}$ on $T_{011}$, $T_{010}$, $T_{001}$ and $T_{000}$ as well as on the complementary intervals $G^3_{01}$, $G^3_{00}$ and $G^3_0$ between them. We let the length of $T_{111}$ be $\frac{1}{3^3}$, the length of $G^3$ be $\frac{1}{3}$, the length of $G^3_{\mathbf{1}|1}$ be $\frac{1}{3^2}$, and the length of $G^3_{\mathbf{1}|2}$ be $\frac{1}{3^3}$. Since $\max T_{000} < 1$, we set $l_3(x) = l_3(\max T_{000})$ for all $x \in (\max T_{000}, 1]$.

At the end of stage 3, sets $\{T_{\mathbf{n}|3} : \mathbf{n} \in \mathcal{N}\}$, $\{G^3_{\mathbf{n}|1} : \mathbf{n} \in \mathcal{N}\}$ and $\{G^3_{\mathbf{n}|2} : \mathbf{n} \in \mathcal{N}\}, G^3$, and a function $l_3$ have been defined with the following properties:

(1) $l_3$ is a $2^3$ function and $l_3$ is piecewise linear,
(2) $sl(l_3) = \frac{1}{2^3}$ for every $x \in T_{111}$,
(3) $sl(l_3) = \frac{1}{2^2}$ for every $x \in T_{110}$,
(4) $sl(l_3) = \frac{1}{2}$ for every $x \in T_{101}$,
(5) $sl(l_3) = \frac{1}{2}$ for every $x \in T_{100}$,
(6) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in T_{011}$,
(7) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in T_{010}$,
(8) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in T_{001}$,
(9) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in T_{000}$,
(10) $|sl(l_3)| < \frac{\frac{1}{2^2}\ \frac{1}{3^2}}{\frac{1}{3^3}}$ for every $x \in G^3_{11}$,
(11) $|sl(l_3)| < \frac{\frac{1}{2}\ \frac{1}{3}}{\frac{1}{3^2}}$ for every $x \in G^3_1$,
(12) $sl(l_3) = \frac{1}{2}$ for every $x \in G^3_{10}$,
(13) $|sl(l_3)| < 3$ for every $x \in G^3$,

(14) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in G^3_{01}$,
(15) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in G^3_{0}$,
(16) $sl(l_3) = 2^{\frac{7}{4}}$ for every $x \in G^3_{00}$,
(17) the length of $F_{111}$ is $\frac{1}{3^3}$, the length of $G^3_{11}$ is $\frac{1}{3^3}$ and the length of $G^3_1$ is $\frac{1}{3^2}$.

At stage $k > 3$ we construct a function $l_k$, sets $\{T_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}$, $\{G^k_{\mathbf{n}|j} : \mathbf{n} \in \mathcal{N}, 1 \leq j \leq k-1\}$, and $G^k$ such that

(1) $l_k$ is a $2^k$ function and $l_k$ is piecewise linear,
(2) $sl(l_k) = \frac{1}{2^k}$ on $T_{\mathbf{1}|k}$,
(3) $sl(l_k) = \frac{1}{2^{j-1}}$ on $T_{\mathbf{n}|k}$ whenever $\mathbf{n}|j = (\mathbf{1}|(j-1), 0)$ as well as on the complementary intervals between them, for $2 \leq j \leq k$,
(4) $sl(l_k) = (2^{2^k-1})^{\frac{1}{2^k-1}}$ on $T_{\mathbf{n}|k}$ whenever $\mathbf{n}|1 = 0$ as well as on the complementary intervals between them, and
(5) $l_k(x) = l_k(\max T_{\mathbf{0}|k})$ for every $x \in (\max T_{\mathbf{0}|k}, 1]$.

Moreover, we let the length of $T_{\mathbf{1}|k}$ be $\frac{1}{3^k}$, the length of $G^k$ be $\frac{1}{3}$ and the length of $G^k_{\mathbf{1}|j}$ be $\frac{1}{3^{j+1}}$ for $1 \leq j \leq k-1$. By construction, $l_k$ is a $2^k$ function for each $k \geq 2$, so that $\{l_k\} \subseteq \mathcal{E}$. To see this, we note that

(1) $l_k(G^k_{\mathbf{n}|j}) = G^k_{A(\mathbf{n})|j}$, for every $2 \leq j \leq k-1$ and $\mathbf{n} \in \mathcal{N}$ such that $\mathbf{n}|j \neq \mathbf{1}|j$,
(2) $\overline{G^k} \subset l_k(G^k)$, and
(3) $\overline{G^k_{\mathbf{0}|j}} \subset l_k(G^k_{\mathbf{1}|j})$, for every $1 \leq j \leq k-1$.

From this we conclude that $l_k^{2^k}$ is linear on $\overline{G^k_{\mathbf{0}|j}}$ with $2 \leq j \leq k-1$ with slope greater than one. Thus, $G^k$ contains exactly one periodic point, which is necessarily a repelling fixed point, each $G^k_{\mathbf{n}|j}$ contains exactly one periodic point of period $2^j$ for every $2 \leq j \leq k-1$ and $\mathbf{n} \in \mathcal{N}$, and these periodic points are also repelling.

Since $l_k$ is linear and orientation-preserving on each $T_{\mathbf{n}|k}$ and $l_k(T_{\mathbf{n}|k}) = T_{A(\mathbf{n})|k}$ for all $\mathbf{n} \in \mathcal{N}$, every point of each $T_{\mathbf{n}|k}$ has period $2^k$. We now show that $\{l_k\}_{k=2}^{\infty}$ is equicontinuous. From our construction, we have that

(1) $sl(l_k) = \frac{1}{2^k}$ on $T_{\mathbf{1}|k}$,
(2) $sl(l_k) = \frac{1}{2^{j-1}}$ on $T_{\mathbf{n}|k}$ whenever $\mathbf{n}|j = (\mathbf{1}|(j-1), 0)$ as well as on the complementary intervals between them, for $2 \leq j \leq k$ , and
(3) $sl(l_k) = (2^{2^k-1})^{\frac{1}{2^k-1}} < 4$ on $T_{\mathbf{n}|k}$ whenever $\mathbf{n}|1 = 0$ as well as on the complementary intervals between them.

Since, for every $1 \leq j \leq k-1$ ,

$$|sl(l_k)| < \frac{\frac{1}{2^j}\frac{1}{3^j}}{\frac{1}{3^{j+1}}} \text{ on } G^k_{\mathbf{1}|j} \tag{1}$$

and

$$|sl(l_k)| < 3 \text{ on } G^k,$$

it follows that for any $k \in \mathbb{N}$ and any $x, y \in [0,1]$,

$$|l_k(x) - l_k(y)| \leq 4|x-y|;$$

hence the sequence $\{l_k\}_{k=2}^{\infty}$ is equicontinuous. Since $l_k$ is a self-map of the unit interval for each $k$, we see that $\{l_k\}_{k=2}^{\infty}$ is also uniformly bounded. By the Ascoli-Arzela Theorem ([12], Theorem 40, page 169), there exists a subsequence of $\{l_k\}_{k=2}^{\infty}$ that converges uniformly to some limit function $l$. Since $\mathcal{E}$ is closed in $\mathcal{C}$, it follows that $h(l) = 0$. For convenience, we shall continue to denote this convergent subsequence as $\{l_k\}_{k=2}^{\infty}$. Since $(\mathcal{K}, \mathcal{H})$ is compact, there exists a subsequence $\{l_{n_k}\}_{k=2}^{\infty}$ of $\{l_k\}_{k=2}^{\infty}$for which the associated sets $\{T_{\mathbf{n}|j} : \mathbf{n} \in \mathcal{N}\}_{j=1}^{2^{n_k}}$ converge in $\mathcal{K}$ to some $\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}_{k=1}^{\infty}$. The collection $\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}_{k=1}^{\infty}$ forms a simple system for $l$, and since $\lim_{k\to\infty} |T_{\mathbf{n}|k}| = 0$ for all $\mathbf{n} \in \mathcal{N}$, by Proposition 2.2 we can conclude that $l$ possesses a unique, and hence perfect, uncountable $\omega$-limit set with respect to $\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}_{k=1}^{\infty}$. We will refer to this $\omega$-limit set as $\Omega'$.

For each $\mathbf{n} \in \mathcal{N}$, let $F_{\mathbf{n}} = \bigcap_{k=1}^{\infty} F_{\mathbf{n}|k} = \bigcap_{k=1}^{\infty}[a_{\mathbf{n}|k}, b_{\mathbf{n}|k}] = b_{\mathbf{n}}$. In the sequel we truncate $l$ at $F_{\mathbf{0}}$, ignoring that part of our graph on the interval $(F_{\mathbf{0}}, 1]$ where $l$ is constant. For convenience we will continue to refer to $l$ as a self-map of $I$ even though it is actually a self-map of the interval $[0, F_{\mathbf{0}}]$ contained in $I$.

In the second part of our construction we modify the simple system $\{F_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}\}_{k=1}^{\infty}$ and $l$ so that we obtain a function $g$ that possesses a unique uncountable $\omega$-limit set with isolated points. Those components of our modified simple system that are nondegenerate will correspond to those elements of $\mathcal{N}$ for which $\mathbf{n}$ terminates in a string of ones. Our new function $g : [-\frac{1}{6}, 1] \to [-\frac{1}{6}, 1]$ will have the following properties:

(1) $h(g) = 0$,
(2) $g$ possesses a unique uncountable $\omega$-limit set with isolated points,
(3) the graph of $g$ is a union of line segments,
(4) each point at which the slope of $g$ changes is isolated, and these points have $0$ and $-\frac{1}{12}$ as their only points of accumulation, and
(5) $g$ is differentiable at $0$ and $-\frac{1}{12}$ with $g'(0) = g'(-\frac{1}{12}) = 0$.

We should stress that our modified function $g$ will differ from $l$ only on the intervals $[-\frac{1}{6}, 0)$, $G$, and $G_{\mathbf{1}|k}$ for $k \geq 1$. The linearity of our original function $l$ on the intervals $F_0, F_{10}, F_{110}, F_{1110}, \ldots$ will be maintained. In fact, it is this monotonicity in conjunction with the adding machine that makes it necessary to modify our original function only on the relatively few intervals listed above.

Our new function $g$ will behave in the following manner on its simple system. Since we will modify $l$ only on $[-\frac{1}{6}, 0)$, $G$, and $G_{\mathbf{1}|k}$ for $k \geq 1$, $\Omega'$ will be invariant. For our new function, $F_{\mathbf{n}}$ will be nondegenerate if and only if $\mathbf{n} \in \mathcal{M} = \{\mathbf{n} \in \mathcal{N} :$ $\mathbf{n}$ terminates with a tail of ones$\}$. For $k \geq 1$, we take the length of $F_{\mathbf{1}|k,0,\mathbf{1}}$ to be $(\frac{1}{4})(\frac{1}{3^{k+1}})$ so that the right endpoint of this nondegenerate component corresponds to the location of $F_{\mathbf{1}|k,0,\mathbf{1}}$ for our original function $l$, and the component $F_{\mathbf{1}|k,0,\mathbf{1}}$ now fills the right quarter of what was $G_{\mathbf{1}|k}$. We let $a_{\mathbf{1}|k,0}$ be the midpoint of $G_{\mathbf{1}|k}$, and, in general, set $a_{\mathbf{1}|k,0,\mathbf{1}|j} = a_{\mathbf{1}|k,0,\mathbf{1}} - (\frac{1}{2^j})(\frac{1}{4})(\frac{1}{3^{k+1}})$. To define $a_{\mathbf{1}|k}$ for $k \geq 1$ and $a_{\mathbf{1}}$ we use the same construction as above, imagining that we have a complementary interval of length $\frac{1}{3}$ to the left of $0$. We set $a_{\mathbf{1}} = -\frac{1}{12}$ and $a_1 = -\frac{1}{6}$ with $a_{\mathbf{1}|k} = -\frac{1}{12} - (\frac{1}{2^{k-1}})(\frac{1}{12})$ for $k \geq 2$. We set $C = \{a_{\mathbf{n}} : \mathbf{n} \in \mathcal{M}\}$ and take $S$ to be all elements of $\Omega'$ except for those that are the right endpoints of an interval complementary to $\Omega'$, and $\{0\}$. If we take $B = \{b_{\mathbf{n}} : \mathbf{n} \in \mathcal{M}\}$, then $\Omega' = S \cup B$. Now, set $L = \Omega' \cup C \cup \{a_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\}$.

If $x \in S$, we define $g(x)$ so that $\{g(x)\} = F_{A(\mathbf{n})}$ where $\{x\} = F_{\mathbf{n}}$. On $C \cup B$ we define $g$ so that $g(a_{\mathbf{n}}) = a_{A(\mathbf{n})}$ and $g(b_{\mathbf{n}}) = b_{A(\mathbf{n})}$, and when $\mathbf{n}|k \neq \mathbf{1}|k$, we let $g(a_{\mathbf{n}|k}) = a_{A(\mathbf{n})|k}$. We complete our description of $g$ on $L$ by setting $g(a_{\mathbf{1}|k}) = a_{\mathbf{0}|(k+1)}$.

How we modify $l$ on $G$ now follows from what we have done on $[-\frac{1}{6}, 0)$ and the adding machine. Since the slope of $l$ on the portion $F_0$ is 4 and $l$ is orientation-preserving there, we need only append a $\frac{1}{4}$-scale copy of $[-\frac{1}{6}, 0)$ to $F_{0\mathbf{1}}$ or the right endpoint of $G$. Thus, our new function $g$ will be linear with slope 4 on $[a_0, F_{\mathbf{0}}]$.

In what follows we assume that the pattern established above for $G_{\mathbf{1}|k}$ and $[-\frac{1}{6}, 0)$ is continued in every interval complementary to $\Omega'$: the nondegenerate portion of our simple system in the arbitrary interval $G_{\mathbf{n}|k}$ will occupy its right fourth with $a_{\mathbf{n}|k,0}$ found at its center, and the point $a_{\mathbf{n}|k,0,\mathbf{1}|j}$ half-way between $a_{\mathbf{n}|k,0,\mathbf{1}|(j-1)}$ and $a_{\mathbf{n}|k,0,\mathbf{1}}$ for $j \geq 2$. The points $b_{\mathbf{n}|k}$ need not change from our original function. As mentioned earlier, however, these modifications are transparent. Since $l$ is linear on every interval $F_{\mathbf{n}|(k-1)}$ such that $\mathbf{n}|(k-1) \neq \mathbf{1}|(k-1)$, the graph of $l$ need not change on any part of the unit interval except on $G$ and $G_{\mathbf{1}|k}$, for all $k$.

We can now take $g$ to be linear on $F_{\mathbf{1}|k,0,\mathbf{1}}$ and $[a_{\mathbf{1}|k,0}, a_{\mathbf{1}|k,0,\mathbf{1}}]$ for $k \geq 1$ since a mildly tedious, but not too difficult, calculation shows that the slope of $g$ on $F_{\mathbf{1}|k,0,\mathbf{1}}$ is less than $\frac{3}{2^k}$ (2) while the slope of $g$ from $a_{\mathbf{1}|k,0}$ to $a_{\mathbf{1}|k,0,\mathbf{1}}$ is less than $\frac{3}{2^{k+1}}$ (3). We note that the slope of $g$ on the rest of the portion $F_{\mathbf{1}|k,0}$, that is, on $[b_{\mathbf{1}|k,0,\mathbf{1}}, b_{\mathbf{1}|k,\mathbf{0}}]$ is still $\frac{1}{2^k}$. We also take $g$ to be linear on $G_{\mathbf{1}|k}$, so that from (1), (2) and (3) we have the absolute value of the slope of $g$ on $G_{\mathbf{1}|k}$ less than $\frac{3}{2^{k-1}}$. To complete our definition of $g$, we extend $g$ in a linear fashion to all of the complementary intervals of $[a_1, a_{\mathbf{1}}]$ and $[a_0, b_{0,\mathbf{1}}]$, and take $g$ to be linear on $G$. As mentioned earlier, this makes $g$ linear with slope 4 on all of $[a_0, F_{\mathbf{0}}]$.

We now verify that

$$\lim_{h\to 0} \frac{g(-\frac{1}{12}+h) - g(-\frac{1}{12})}{h} = 0.$$

This is clear as $h \to 0^+$ since the entire component $F_{\mathbf{1}} = [-\frac{1}{12}, 0]$ is mapped to the point $F_{\mathbf{0}}$. As for $h$ approaching 0 from the left, we have

$$\lim_{h\to 0^-} \frac{g(-\frac{1}{12}+h) - g(-\frac{1}{12})}{h} \geq 0,$$

and since

$$\frac{g(-\frac{1}{12}+h) - g(-\frac{1}{12})}{h} \leq \frac{(\frac{1}{3^k})(\frac{1}{2^k})}{(\frac{1}{2^k})(\frac{1}{12})}$$

whenever $-h < (\frac{1}{2^{k-1}})(\frac{1}{12})$, our conclusion follows.

Our next task is to verify that $g : [-\frac{1}{6}, 1] \to [-\frac{1}{6}, 1]$ has zero topological entropy. From our development of $g$, one sees that for every $k \geq 1$,

(1) $g(G_{\mathbf{n}|k}) = G_{A(\mathbf{n})|k}$, for $\mathbf{n}|k \neq \mathbf{1}|k$,
(2) $\overline{G} \subset g(G)$, and
(3) $\overline{G_{\mathbf{0}|k}} \subset g(G_{\mathbf{1}|k})$.

From this we conclude that $g^{2^k}$ is linear on $G_{\mathbf{0}|k}$ and has slope greater than one. Thus, $G$ contains exactly one periodic point which is necessarily a repelling fixed point, and each $G_{\mathbf{n}|k}$ contains exactly one periodic point of period $2^k$, which is also repelling (4). We also note that $g(F_{\mathbf{n}|k}) = F_{A(\mathbf{n})|k}$ whenever $\mathbf{n}|k \neq \mathbf{1}|k$, and $g(F_{\mathbf{1}|k})$ is a proper subset of $F_{\mathbf{0}|k}$ (5). From properties (4) and (5), arguing as in in the proof of Theorem 4.3 in [6], we can conclude that for each $x \in [-\frac{1}{6}, 1]$, either $\omega(x, f)$ is a $2^k$ cycle for some $k$, or $\omega(x, f)$ is contained in $\bigcap_{k=1}^{\infty} \bigcup_{\mathbf{n}\in\mathcal{N}} F_{\mathbf{n}|k}$. Since

$\bigcap_{k=1}^{\infty} \bigcup_{\mathbf{n}\in\mathcal{N}} F_{\mathbf{n}|k}$ contains no cycles, it follows that $g$ must be a $2^{\infty}$ function and that $h(g) = 0$.

That $g$ possesses an uncountable $\omega$-limit set with isolated points follows from the observation that the orbit of $a_0$ is $\{a_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N} \text{ and } k \in \mathbb{N}\}$ so that $\omega(a_0, f) = \Omega' \cup C$. Since all the points of $C$ are contained in a single orbit, $\Omega' \cup C$ is the unique uncountable $\omega$-limit of $g$ with isolated points.

In this third part of our construction we modify the function $g : [-\frac{1}{6}, 1] \to [-\frac{1}{6}, 1]$ so that it is continuously differentiable there. Loosely speaking, we need to round the corners of $g$ at

(1) $\inf G$ and $\sup G = a_0$,
(2) $\inf G_{\mathbf{1}|k}$, $\sup G_{\mathbf{1}|k} = a_{\mathbf{1}|k,0}$, $a_{\mathbf{1}|k,0,\mathbf{1}}$ and $b_{\mathbf{1}|k,0,\mathbf{1}}$ for every $k \geq 1$, and
(3) $a_{\mathbf{1}|k}$ for $k \geq 1$.

In doing so we must take care to ensure that the topological entropy of our continuously differentiable function $f$ is zero.

Prior to making our final set of modifications it may be helpful to recall the slope of $g$ on some subintervals of $[-\frac{1}{6}, 1]$. In particular,

(1) $sl(g) = 4$ on $[a_0, F_{\mathbf{0}}]$,
(2) $sl(g) = \frac{1}{2^k}$ on $[b_{\mathbf{1}|k,0,\mathbf{1}}, b_{\mathbf{1}|k,0}]$,
(3) $sl(g) < \frac{3}{2^k}$ on $F_{\mathbf{1}|k,0,\mathbf{1}}$,
(4) $sl(g) < \frac{3}{2^{k+1}}$ on $[a_{\mathbf{1}|k,0}, a_{\mathbf{1}|k,0,\mathbf{1}}]$, and
(5) $-\frac{3}{2^{k-1}} < sl(g) < 0$ on $G_{\mathbf{1}|k}$.

We begin by modifying $g$ so that it is differentiable at $\sup G$ and $\inf G$. Let $D_0 = (\frac{1}{3})(\frac{1}{2^5})$, set $z_0 = \inf G + D_0$ and pick $x_0 \in G$ so that

$$g(x_0) < a_{0,1} \quad \text{and} \quad M_0^* = \frac{g(\inf G) - g(x_0)}{z_0 - x_0} > 2M_0,$$

where $M_0$ is the slope of $g$ on $G$. We modify $g$ on $[\inf G, a_{0,1}]$ in the following ways.

(1) On $[a_0, a_{0,1}]$: We define $f(a_0) = g(a_0)$ and $f(a_{0,1}) = g(a_{0,1})$, and perturb $g$ so that $f^{'}(a_{0,1}) = 4$, $f'_+(a_0) = 0$ and, if $x \in (a_0, a_{0,1})$, then $0 < f'(x) < 8$.
(2) On $[x_0, a_0)$: We define $f(x_0) = g(x_0)$, and perturb $g$ so that $f^{'}(x_0) = M_0^*$, $f'_-(a_0) = 0$ and, if $x \in (x_0, a_0)$, then $2M_0^* < f'(x) < 0$.
(3) On $[z_0, x_0)$: We set $f(z_0) = g(\inf G)$ and we make $f$ linear with slope $M_0^*$ from $(z_0, f(z_0))$ to $(x_0, f(x_0))$.
(4) On $[\inf G, z_0)$: We define $f(\inf G) = g(\inf G)$ and, calling $y$ the midpoint of $[\inf G, z_0]$, we set $f(y) = (\frac{1}{12})(\frac{1}{2})D_0 + f(\inf G)$, and we take $f$ so that $f'(\inf G) = \frac{1}{2}$, $f'(z_0) = M_0^*$, $f'(y) = 0$, and, if $x \in (\inf G, y)$, then $0 < f'(x) < \frac{1}{2}$, and if $x \in (y, z_0)$, then $M_0^* < f'(x) < 0$.

We now modify $g$ so that $f$ is differentiable at $\sup G_{\mathbf{1}|k} = a_{\mathbf{1}|k,0}$, $a_{\mathbf{1}|k,0,\mathbf{1}}$, and $b_{\mathbf{1}|k,0,\mathbf{1}}$ for every $k \geq 1$. In $G_{\mathbf{1}|k}$ we let $z_{\mathbf{1}|k} = \inf G_{\mathbf{1}|k} + (\frac{1}{3^k})(\frac{1}{2^k})D_0$ and pick $x_{\mathbf{1}|k}$ so that

$$f(x_{\mathbf{1}|k}) < a_{\mathbf{1}|k,0,1} \quad \text{and} \quad M_{\mathbf{1}|k}^* = \frac{g(\inf G_{\mathbf{1}|k}) - g(x_{\mathbf{1}|k})}{z_{\mathbf{1}|k} - x_{\mathbf{1}|k}} > -\frac{3}{2^{k-2}}.$$

Since the slope of $g$ on $G_{\mathbf{1}|k}$ is greater than $-\frac{3}{2^{k-1}}$, this restriction is not difficult to satisfy. We now modify $g$ on $[\inf G_{\mathbf{1}|k}, b_{\mathbf{1}|k,0,\mathbf{1}}]$ in the following ways.

(1) On $[a_{\mathbf{1}|k,0,\mathbf{1}}, b_{\mathbf{1}|k,0,\mathbf{1}}]$: We define $f(a_{\mathbf{1}|k,0,\mathbf{1}}) = g(a_{\mathbf{1}|k,0,\mathbf{1}})$ and $f(b_{\mathbf{1}|k,0,\mathbf{1}}) = g(a_{\mathbf{1}|k,0,\mathbf{1}})$, and perturb $g$ so that $f^{'}(a_{\mathbf{1}|k,0,\mathbf{1}}) = r$, where $r$ is the slope of

$g$ on $[a_{\mathbf{1}|k,0}, a_{\mathbf{1}|k,0,\mathbf{1}}]$, $f^{'}(b_{\mathbf{1}|k,0,\mathbf{1}}) = \frac{1}{2^k}$ and, if $x \in (a_{\mathbf{1}|k,0,\mathbf{1}}, b_{\mathbf{1}|k,0,\mathbf{1}})$, then $\frac{1}{2^k} < f'(x) < 4r$.

(2) On $[a_{\mathbf{1}|k,0}, a_{\mathbf{1}|k,0,1}]$: We define $f(a_{\mathbf{1}|k,0}) = g(a_{\mathbf{1}|k,0})$, and perturb $g$ so that $f_{+}^{'}(a_{\mathbf{1}|k,0}) = 0$, and, if $x \in (a_{\mathbf{1}|k,0}, a_{\mathbf{1}|k,0,1})$, then $0 < f'(x) < 2r$. We also define $f$ to be equal to $g$ on $a_{\mathbf{1}|k,0,1}$.

(3) On $[x_{\mathbf{1}|k}, a_{\mathbf{1}|k,0})$: We define $f(x_{\mathbf{1}|k}) = g(x_{\mathbf{1}|k})$, and perturb $g$ so that $f^{'}(x_{\mathbf{1}|k}) = M^*_{\mathbf{1}|k}$ and $f_{-}^{'}(a_{\mathbf{1}|k,0}) = 0$ and, if $x \in (x_{\mathbf{1}|k}, a_{\mathbf{1}|k,0})$, then $2M^*_{\mathbf{1}|k} < f'(x) < 0$.

(4) On $[z_{\mathbf{1}|k}, x_{\mathbf{1}|k})$: We define $f(z_{\mathbf{1}|k}) = g(\inf G_{\mathbf{1}|k})$, and we make $f$ linear with slope $M^*_{\mathbf{1}|k}$ from $(z_{\mathbf{1}|k}, f(z_{\mathbf{1}|k}))$ to $(x_{\mathbf{1}|k}, f(x_{\mathbf{1}|k}))$.

(5) On $[\inf G_{\mathbf{1}|k}, z_{\mathbf{1}|k})$: We define $f(\inf G_{\mathbf{1}|k}) = g(\inf G_{\mathbf{1}|k})$, and, calling $y$ the midpoint of $[\inf G_{\mathbf{1}|k}, z_{\mathbf{1}|k}]$, we set $f(y) = (\frac{1}{12})(\frac{1}{2^{k+1}})[(\frac{1}{3^k})(\frac{1}{2^k})D_0] + f(\inf G_{\mathbf{1}|k})$, and we take $f$ so that $f'(\inf G_{\mathbf{1}|k}) = \frac{1}{2^{k+1}}$, $f'(z_{\mathbf{1}|k}) = M^*_{\mathbf{1}|k}$, $f'(y) = 0$, and, if $x \in (\inf G_{\mathbf{1}|k}, y)$, then $0 < f'(x) < \frac{1}{2^{k+1}}$ and, if $x \in (y, z_{\mathbf{1}|k})$, then $M^*_{\mathbf{1}|k} < f'(x) < 0$.

The last interval on which we need to modify $g$ is $[a_1, a_{\mathbf{1}}]$. There we set $f(a_1) = g(a_1)$, $f(a_{\mathbf{1}|k}) = g(a_{\mathbf{1}|k})$ for $k \geq 1$ and $f(a_{\mathbf{1}}) = g(a_{\mathbf{1}})$ and perturb $g$ so that $f'_{-}(a_{\mathbf{1}}) = 0$, $f'$ is continuously differentiable on $[a_1, a_{\mathbf{1}}]$, and $f'' \leq 0$ there. Since $a_{\mathbf{1}} = -\frac{1}{12}$, $a_1 = -\frac{1}{6}$ and, in general, $a_{\mathbf{1}|k} = -\frac{1}{12} - (\frac{1}{2^{k-1}})(\frac{1}{12})$ for $k \geq 2$ with $f(a_{\mathbf{1}|k}) = a_{\mathbf{0}|(k+1)}$ and $f(a_{\mathbf{1}}) = F_{\mathbf{0}}$, the existence of such a perturbation of $g$ is assured.

Since the behavior of $g$ on $L$ is unchanged, the orbit of $a_0$ is still $\{a_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}\}$, so that $\omega(a_0, f) = \Omega' \cup C$, as before.

Our function $f$ is also continuously differentiable. That $f$ is differentiable at every point of $[-\frac{1}{6}, 1]\backslash\{0\}$ follows from the modifications just prescribed. By bounding the slopes of the perturbations as we did, it also follows that $\lim_{x\to 0} f'(x) = 0 = f'(0)$. Since zero is the unique accumulation point of the "corners" found at $\sup G$, $\inf G$, and the set $\{\sup G_{\mathbf{1}|k}, \inf G_{\mathbf{1}|k}, a_{\mathbf{1}|k,0,\mathbf{1}}, b_{\mathbf{n}|k,0,\mathbf{1}} : k \geq 1\}$, we conclude that $f$ is continuously differentiable.

We now verify that $h(f) = 0$.

We begin by considering the intervals $[\inf G, z_0]$ and $[\inf G_{\mathbf{1}|k}, z_{\mathbf{1}|k}]$, for $k \geq 1$. By choosing $\{z_{\mathbf{1}|k}\}_{k=1}^{\infty}$ and defining $f$ on $Z = [\inf G, z_0] \cup \bigcup_{k=1}^{\infty} [\inf G_{\mathbf{1}|k}, z_{\mathbf{1}|k}]$ as we did, we have $f^2([\inf G, z_0]) \subseteq [\inf G_1, z_1]$ and, in general, $f^{2^{k+1}}([\inf G_{\mathbf{1}|k}, z_{\mathbf{1}|k}]) \subseteq [\inf G_{\mathbf{1}|(k+1)}, z_{\mathbf{1}|(k+1)}]$. Thus, $\omega(x, f) = \omega(\inf G_{\mathbf{1}|k}, f) = Q$ for every $x$ contained in $Z$.

Since $g$ was not affected on the intervals $G_{\mathbf{n}|k}$ for $\mathbf{n}|k \neq \mathbf{1}|k$, as before $f(G_{\mathbf{n}|k}) = G_{A(\mathbf{n})|k}$. We also have $\overline{G_{\mathbf{0}|k}} \subset f(G_{\mathbf{1}|k})$ with $f$ linear on $f^{-1}(\overline{G_{\mathbf{0}|k}}) \cap G_{\mathbf{1}|k}$. From this we conclude that each $G_{\mathbf{n}|k}$ contains exactly one periodic point of period $2^k$, which must be repelling. Similarly, $\overline{G} \subset f(G)$ and $f$ linear on $f^{-1}(\overline{G}) \cap G$ imply that the unique periodic point contained in $G$ is a repelling fixed point. As before, $f(F_{\mathbf{n}|k}) = F_{A(\mathbf{n})|k}$ whenever $\mathbf{n}|k \neq \mathbf{1}|k$ with $f(F_{\mathbf{1}|k}) \subset F_{\mathbf{0}|k}$. Arguing as in the proof of Theorem 4.3 in [6] we can conclude that for each $x \in [-\frac{1}{6}, 1]$, either $\omega(x, f)$ is a $2^k$ cycle for some $k$, or $\omega(x, f)$ is contained in $\bigcap_{k=1}^{\infty} \bigcup_{\mathbf{n}\in\mathcal{N}} F_{\mathbf{n}|k}$. Since $\bigcap_{k=1}^{\infty} \bigcup_{\mathbf{n}\in\mathcal{N}} F_{\mathbf{n}|k}$ contains no cycles, it follows that $f$ must be a $2^{\infty}$ function and that $h(f) = 0$.

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